

# GEOMETRY AND SPECTRUM OF RAPIDLY BRANCHING GRAPHS

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**ABSTRACT.** We study graphs whose vertex degree tends and which are, therefore, called rapidly branching. We prove spectral estimates, discreteness of spectrum, first order eigenvalue and Weyl asymptotics solely in terms of the vertex degree growth. The underlying techniques are estimates on the isoperimetric constant. Furthermore, we give lower volume growth bounds and we provide a new criterion for stochastic incompleteness.

## 1. INTRODUCTION

The spectral theory of Laplacians on graph is a vibrant topic of study. A specific focus lies on geometric criteria for spectral bounds and discreteness of spectrum, see e.g. [Dod84, DK86, BGK13, Fuj96, Gol14, Kel10, KL10, KPP, Moh91, Woj09, Žuk97]. A necessary condition for discreteness of spectrum is that the vertex degree tends to infinity, a condition which is called rapidly branching, [Fuj96]. This condition is by no means sufficient. Throughout the years various geometric criteria were given to ensure that rapid branching implies purely discrete spectrum, see e.g. [BGK13, Fuj96, Kel10, KPP, Woj09].

The novelty of this work is to provide one single and concise criterion on the growth of the vertex degree which by itself implies discreteness of spectrum. Specifically, we consider graphs whose vertex degrees grow proportionally to the numbers of vertices with smaller degree. In this sense we control the acceleration of the vertex degree growth. We discover that for these graphs, there is no need to impose a priori conditions on the underlying geometry such as planarity or curvature assumptions. Furthermore, we derive valuable geometric implications such as bounds for the isoperimetric constant at infinity and volume growth bounds.

Moreover, we study stochastic completeness which is a topic that has also been investigated intensively for graphs with unbounded degree throughout the recent years, cf. e.g. [Fol14a, GHM12, Hua11b,

[Hua14, KL12, KL10, K LW13, Woj08, Woj09, Woj11]. We show that, for a certain acceleration of the vertex degree growth, the graphs are stochastically incomplete.

**1.1. Set-up and definitions.** Let  $G = (V, E)$  be an infinite, simple, locally finite, connected graph. The vertex degree  $\deg : V \rightarrow \mathbb{N}$  assigns to each vertex  $v$  the number of edges emanating from  $v$ . Furthermore, denote  $\eta : \mathbb{R} \rightarrow \mathbb{N} \cup \{\infty\}$

$$\eta(k) = \#\{v \in V \mid \deg(v) \leq k\}.$$

We are interested in graphs for which  $\eta$  grows very slowly. That is for each bound  $k$  there are only few vertices with degree smaller than  $k$ . This is measured by the following constants

$$r = \sup_{k \in \mathbb{N}} \frac{\eta(k)}{k} \quad \text{and} \quad r_\infty = \limsup_{k \rightarrow \infty} \frac{\eta(k)}{k}.$$

Clearly,  $0 \leq r_\infty \leq r \leq \infty$ . We can think of  $1/r$  or  $1/r_\infty$  as a type of acceleration rate. One can also see  $\eta$  as the spectral counting function of the multiplication operator with the vertex degree function.

Let

$$d = \min_{v \in V} \deg(v) \quad \text{and} \quad d_\infty = \sup_{K \subseteq V \text{ finite}} \min_{v \in V \setminus K} \deg(v).$$

In [Fuj96, Kel10] graphs with  $d_\infty = \infty$  are referred to as *rapidly branching*. In this case we enumerate the vertices  $V = \{v_k\}_{k \geq 0}$  such that

$$\deg(v_k) \leq \deg(v_{k+1})$$

and define

$$d_k = \deg(v_k)$$

for  $k \geq 0$ . Obviously,  $d \leq d_k \nearrow d_\infty$  as  $k \rightarrow \infty$ .

The main focus of this paper is the spectral theory of the Laplacian

$$\Delta \varphi(v) = \sum_{w \sim v} (\varphi(v) - \varphi(w))$$

which is a positive selfadjoint operator on the Hilbert space  $\ell^2(V)$  of real valued square summable functions with domain

$$D(\Delta) = \{\varphi \in \ell^2(V) \mid \Delta \varphi \in \ell^2(V)\},$$

see [Woj08, Section 1.3]. We denote the bottom of the spectrum of  $\Delta$  by  $\lambda_0 = \lambda_0(\Delta)$ . We say that the spectrum of  $\Delta$  is *purely discrete* if it consists only of isolated eigenvalues of finite multiplicity. It is easy to see that a necessary condition for purely discrete spectrum is  $d_\infty = \infty$ ,

see e.g. [Kel10, Proposition 5]. An example to see that this condition is not sufficient can be found in [Kel10, Theorem 6.1].

In the case of purely discrete eigenvalues we enumerate the eigenvalues of  $\Delta$  in increasing order and counted with multiplicity by  $\lambda_k$ ,  $k \geq 0$ . Moreover, we denote the eigenvalue counting function of  $\Delta$  by

$$N(\lambda) = \sup\{k \geq 0 \mid \lambda_k \leq \lambda\},$$

where  $\sigma(\Delta)$  is the spectrum of  $\Delta$ .

**1.2. Results.** First we present the results on the spectral theory of  $\Delta$ . We denote

$$\gamma(s) = \sqrt{1 - \frac{(1-s)^4}{(1+s^2)^2}}$$

for  $s \in [0, 1]$  and notice that  $\gamma(s) \in [0, 1]$  while  $\gamma(0) = 0$  and  $\gamma(1) = 1$ .

The first result deals with the spectral gap.

**Theorem 1.1.** *If  $d_\infty = \infty$  and  $r_\infty < 1$ , then  $\lambda_0 > 0$ . If even  $r < 1$ , then*

$$d(1 - \gamma(r)) \leq \lambda_0.$$

Secondly, we give a criterion for discreteness of the spectrum. Furthermore, we provide estimates on the corresponding Weyl and eigenvalue asymptotics.

**Theorem 1.2.** *If  $d_\infty = \infty$  and  $r_\infty < 1$ , then  $\Delta$  has purely discrete spectrum and we have*

$$1 - \gamma(r) \leq \liminf_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\eta(\lambda)} \leq \limsup_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\eta(\lambda)} \leq 1 + \gamma(r)$$

and

$$1 - \gamma(r_\infty) \leq \liminf_{k \rightarrow \infty} \frac{\lambda_k}{d_k} \leq \limsup_{k \rightarrow \infty} \frac{\lambda_k}{d_k} \leq 1 + \gamma(r_\infty).$$

*In particular, if  $r_\infty = 0$ , then  $\lambda_k/d_k \rightarrow 1$  as  $k \rightarrow \infty$ .*

The results of the theorems above are based on the following estimate of the isoperimetric constant  $\alpha$  and the isoperimetric constant at infinity  $\alpha_\infty$  defined as

$$\alpha = \inf_{U \subseteq V \text{ finite}} \frac{\#\partial U}{\text{vol}(U)} \quad \text{and} \quad \alpha_\infty = \sup_{K \subseteq V \text{ finite}} \inf_{U \subseteq V \setminus K \text{ finite}} \frac{\#\partial U}{\text{vol}(U)},$$

where  $\partial U = \{(v, w) \in U \times (V \setminus U) \mid v \sim w\}$  and  $\text{vol}(U) = \sum_{v \in U} \deg(v)$ .

**Theorem 1.3.** *Assume  $d_\infty = \infty$ . Then,*

$$\alpha \geq 1 - \frac{2r}{1+r^2} \quad \text{if } r \leq 1 \quad \text{and}$$

$$\alpha_\infty \geq 1 - \frac{2r_\infty}{1+r_\infty^2} \quad \text{if } r_\infty \leq 1.$$

The proofs of these theorems are found in Section 2. Let us mention that the results above are sharp in the sense that for given  $s \in (0, 1)$  there are graphs  $G_s$  such that  $s = r_\infty$  and such that the inequality for  $\alpha_\infty$  is an equality.

**Remark 1.4.** Next to the operator  $\Delta$ , one often considers the normalized Laplacian  $\Delta_n$  that is the bounded, positive, selfadjoint operator acting as

$$\Delta_n \varphi(v) = \frac{1}{\deg(v)} \sum_{w \sim v} (\varphi(v) - \varphi(w))$$

on  $\ell^2(V, \deg)$ . For example this operator is studied in [DK86, Fuj96]. By the considerations of [Fuj96] and Theorem 1.3 above, we obtain the spectral estimates  $\lambda_0(\Delta_n) \geq \gamma(r)$  and  $\lambda_0^{\text{ess}}(\Delta_n) \geq 1 - \gamma(r_\infty)$ , where  $\lambda_0(\Delta_n)$  and  $\lambda_0^{\text{ess}}(\Delta_n)$  is the bottom of the spectrum and the bottom of the essential spectrum of  $\Delta_n$ . Furthermore, if  $r_\infty = 0$ , then the essential spectrum of  $\Delta_n$  is equal to  $\{1\}$ .

Next to isoperimetric estimates, we show a lower exponential volume growth bound for graphs with  $r_\infty < 1$ . For a given vertex  $v \in V$ , we denote by  $B_n$  the vertices which can be connected to  $v$  by a path of less or equal to  $n$  edges.

**Theorem 1.5.** *Assume  $d_\infty = \infty$  and  $0 < r_\infty < 1$ . Let  $a$  be the largest real root of the polynomial*

$$p(z) = z^3 - r_\infty^{-1} z^2 - 1.$$

*Then,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{vol}(B_n) \geq 2 \log a.$$

*In particular,  $a \geq r_\infty^{-1}$ .*

This result is proven in Section 3. Of course, there are many examples of graphs with  $r_\infty > 0$  with superexponential volume growth such as trees. However, in Section 3 we present also an example showing that the result is sharp.

Finally, we turn to a property of graphs called stochastic completeness. In the discrete setting the investigation of this topic goes back

to Feller [Fel58, Fel57] and Reuter [Reu57]. Recent interest in this topic was sparked by the Ph.D. thesis of Wojciechowski [Woj08], see also [GHM12, Hua11a, Hua14, Fol14b, K LW13, Woj09, Woj11] and references therein.

A graph is said to be *stochastically complete* if

$$e^{-t\Delta}1 = 1, \quad t \geq 0,$$

where  $e^{-t\Delta}$  is the  $\ell^2$  semigroup of  $\Delta$  which is extended to the bounded functions via monotone limits and 1 denotes the function that is constantly one on  $V$ . For details, see [Woj08]. Note that by general theory one always has  $e^{-t\Delta} \leq 1$  and a graph is said to be *stochastically incomplete* if

$$e^{-t\Delta}1 < 1, \text{ for some (all) } t > 0.$$

The importance of this property stems from the fact that stochastic completeness is equivalent to uniqueness of bounded solutions to the heat equation, see [Woj08, Theorem 3.1.3]. Intuitively, stochastic incompleteness can be understood that heat vanishes from the graph in finite time.

Here, we give a new criterion for stochastic incompleteness.

**Theorem 1.6.** *Let  $d_\infty = \infty$ . If  $r_\infty < e^{-1}$ , then the graph is stochastically incomplete.*

The constant  $e$  in the theorem denotes the Euler number. The proof is given in Section 4.

The authors conjecture that the statement of the theorem remains true for  $r_\infty < 1$ . However, the idea of proof given here does not extend to this situation.

## 2. ISOPERIMETRIC CONSTANTS

In this section we show Theorem 1.3 from which we deduce Theorem 1.1 and Theorem 1.2.

We start with a basic but important fact which will be used successively throughout the paper. Recall that we enumerated the vertices  $V = \{v_k\}_{k \geq 0}$  with respect to increasing vertex degree.

**Lemma 2.1.** *For  $s > r_\infty$  there is  $n \geq 0$  such that for  $k \geq n$  we have*

$$\frac{(k+1)}{s} \leq \deg(v_k).$$

*Proof.* By the inclusion  $\{v_0, \dots, v_k\} \subseteq \{v \in V \mid \deg(v) \leq \deg(v_k)\}$  we have

$$(k+1) \leq \eta(\deg(v_k)).$$

Let now  $n$  be such that  $\eta(\deg(v_k)) \leq s \deg(v_k)$  for  $k \geq n$ . Combining this with the inequality above yields the statement.  $\square$

Now, we give the proof of Theorem 1.3. We recall the notation  $a \wedge b = \min\{a, b\}$ ,  $a \vee b = \max\{a, b\}$  and  $a_+ = a \vee 0$  for numbers  $a, b \in \mathbb{R}$ .

*Proof of Theorem 1.3.* Let  $U \subseteq V$  and let  $N \geq 0$  be such that  $\#U = N + 1$ . Since every vertex in  $U$  can be connected to at most  $N$  vertices within  $U$ , we estimate

$$\#\partial U \geq \sum_{v \in U} (\deg(v) - N)_+ = -N(N + 1) + \sum_{v \in U} (\deg(v) \vee N),$$

and

$$\text{vol}(U) \leq \sum_{v \in U} (\deg(v) \vee N).$$

Hence,

$$\frac{\#\partial U}{\text{vol}(U)} \geq 1 - \frac{N(N + 1)}{\sum_{v \in U} (\deg(v) \vee N)}.$$

For  $s \in (r_\infty, 1]$ , let  $n \geq 1$  be chosen according to Lemma 2.1. For  $K = \{v_0, \dots, v_{n-1}\}$  and  $U \subseteq V \setminus K$  with  $\#U = N + 1$ , we conclude by Lemma 2.1

$$\begin{aligned} \sum_{v \in U} (\deg(v) \vee N) &\geq \sum_{k=n}^{n+N} (\deg(v_k) \vee N) \geq \sum_{k=n}^{n+N} \left( \frac{(k+1)}{s} \vee N \right) \\ &\geq \sum_{k=1}^{N+1} \left( \frac{k}{s} \vee N \right) \geq \frac{1}{s} \int_0^{N+1} (k \vee sN) dk \\ &= \frac{1}{s} \left( (sN)^2 + \frac{1}{2} ((N+1)^2 - (sN)^2) \right) \\ &\geq \frac{s^2 + 1}{2s} N(N + 1). \end{aligned}$$

Plugging this in the inequality for  $\#\partial U / \text{vol}(U)$  above, we conclude the statement for  $\alpha_\infty$ . The corresponding statement for  $\alpha$  follows analogously by letting  $s = r$  and  $n = 0$ .  $\square$

The proofs of Theorem 1.1 and Theorem 1.2 are based on Cheeger estimates for which there is a huge body of literature, see e.g. [Dod84, DK86, Fuj96, Moh88, Moh91, KL10]. We denote the functions of finite support on  $V$  by  $C_c(V)$  and denote the scalar product of  $\ell^2(V)$  by  $\langle \cdot, \cdot \rangle$ .

*Proof of Theorem 1.1 and Theorem 1.2.* The following inequality can be directly extracted from [KL10, Proof of Proposition 15]

$$(1 - \sqrt{1 - \alpha^2}) \langle \deg \varphi, \varphi \rangle \leq \langle \Delta \varphi, \varphi \rangle \leq (1 + \sqrt{1 - \alpha^2}) \langle \deg \varphi, \varphi \rangle,$$

for all  $\varphi \in C_c(V)$ . This yields the bound in Theorem 1.1. Furthermore, from [BGK13, Theorem 5.3] it follows that for all  $\varepsilon > 0$  there is  $C_\varepsilon > 0$  such that for all normalized functions  $\varphi$  with finite support

$$\begin{aligned} (1 - \varepsilon)(1 - \sqrt{1 - \alpha_\infty^2}) \langle \deg \varphi, \varphi \rangle - C_\varepsilon &\leq \langle \Delta \varphi, \varphi \rangle \\ &\leq (1 + \varepsilon)(1 + \sqrt{1 - \alpha_\infty^2}) \langle \deg \varphi, \varphi \rangle + C_\varepsilon. \end{aligned}$$

By the Min-Max-Principle [RS78, Chapter XIII.1] (confer [BGK13, Theorem A.2] or [Gol14] for the details of the application) we deduce the statement about discreteness of spectrum as well as the Weyl and eigenvalue asymptotics. This proves Theorem 1.2. Finally, as the spectrum is purely discrete if  $r_\infty < 1$ , we deduce that  $\lambda_0 > 0$ . To see this, assume for a moment that  $\lambda_0 = 0$  is an eigenvalue. Noting that

$$0 = \langle \Delta \varphi, \varphi \rangle = \frac{1}{2} \sum_{x \sim y} (\varphi(x) - \varphi(y))^2$$

for the underlying eigenfunction  $\varphi$  shows that  $\varphi$  must be a constant function. However, the only constant function in  $\ell^2(V)$  is the zero function. Hence,  $\varphi$  is not an eigenfunction and 0 cannot be an eigenvalue. This finishes the proof of Theorem 1.1.  $\square$

Let us turn to an example which shows that the bound in Theorem 1.3 above is sharp.

**Example 2.2.** We construct a graph for given  $s \in (0, 1)$  as follows. Let  $K_n = (V_n, E_n)$  be complete graphs with  $l(n)$  vertices, where  $l(0) = 2$  and

$$l(n+1) = \lceil s^{-1} \rceil l(n)^2, \quad n \geq 0.$$

Enumerate the vertices in  $K_n$  by  $v_1^{(n)}, \dots, v_{l(n)}^{(n)}$  and connect  $v_j^{(n)}$  with exactly  $(\lceil js^{-1} \rceil - l(n))_+$  vertices in  $K_{n+1}$  such that every vertex in  $K_{n+1}$  is connected to at most one vertex in  $K_n$ . This is possible since

$$\sum_{j=1}^{l(n)} (\lceil js^{-1} \rceil - l(n))_+ \leq \lceil s^{-1} \rceil l(n)^2 = l(n+1).$$

We denote the resulting graph by  $G_s = (V, E)$ . We show that

$$\limsup_{k \rightarrow \infty} \frac{\eta(k)}{k} = s \quad \text{and} \quad \alpha_\infty = 1 - \frac{2s}{1 + s^2}.$$

First, we observe that it suffices to show

$$\limsup_{k \rightarrow \infty} \frac{\eta(k)}{k} \leq s \quad \text{and} \quad \alpha_\infty \leq 1 - \frac{2s}{1 + s^2}.$$

Indeed, if  $\tilde{s} = \limsup_{k \rightarrow \infty} \frac{\eta(k)}{k} \leq s$ , then by Theorem 1.3 we infer

$$1 - \frac{2s}{1 + s^2} \leq 1 - \frac{2\tilde{s}}{1 + \tilde{s}^2} \leq \alpha_\infty \leq 1 - \frac{2s}{1 + s^2}.$$

This implies  $\tilde{s} = s$  and  $\alpha_\infty = 1 - 2s/(1 + s^2)$ . For the vertices  $v_1^{(n)}, \dots, v_{l(n)}^{(n)}$  in  $V_n$ , we observe for the vertex degrees in  $G_s$

$$\lceil js^{-1} \rceil \vee l(n) - 1 \leq \deg(v_j^{(n)}) \leq \lceil js^{-1} \rceil \vee l(n)$$

since there are  $l(n) - 1$  neighbors in  $V_n$ ,  $(\lceil js^{-1} \rceil - l(n))_+$  in  $V_{n+1}$  and 0 or 1 neighbor in  $V_{n-1}$ .

Let now  $k \geq 2$ . If  $\deg(v_j^{(n)}) \leq k - 1$ , we have  $\lceil js^{-1} \rceil \vee l(n) \leq k$ . This implies  $j \leq ks$ . Note further that there is a unique number  $N = N_k \in \mathbb{N}$  such that  $l(N_k) \leq k < l(N_k + 1)$ . It follows that  $n \leq N_k$ . Hence, for the vertices  $v_1^{(N)}, \dots, v_{l(N)}^{(N)} \in V_N$ , we deduce

$$\{v \in V \mid \deg(v) \leq k - 1\} \subseteq \{v_j^{(N)} \mid j \leq ks\} \cup \bigcup_{n=1}^{N_k-1} V_n.$$

Thus, using the inequality  $2l(n) \leq l(n + 1)$  iteratively, yields

$$\eta(k - 1) \leq ks + \sum_{n=1}^{N_k-1} l(n) \leq ks + 2l(N_k - 1).$$

We estimate, using  $k \geq l(N_k)$  and  $l(N_k) = \lceil s^{-1} \rceil l(N_k - 1)^2$ ,

$$\frac{\eta(k - 1)}{k} \leq \frac{ks + 2l(N_k - 1)}{k} \leq s + 2 \frac{l(N_k - 1)}{l(N_k)} = s + 2 \frac{1}{\lceil s^{-1} \rceil l(N_k - 1)},$$

and conclude

$$\limsup_{k \rightarrow \infty} \frac{\eta(k)}{k} \leq s.$$



To show  $\alpha_\infty \leq 1 - 2s/(1 + s^2)$ , we consider  $\#\partial V_n/\text{vol}(V_n)$ . We start by estimating

$$\begin{aligned} \text{vol}(V_n) &= \sum_{j=1}^{l(n)} \deg(v_j^{(n)}) \leq \sum_{j=1}^{l(n)} ([js^{-1}] \vee l(n)) = \int_0^{l(n)} ([[j]s^{-1}] \vee l(n)) \, dj \\ &\leq 2s^{-1}l(n) + s^{-1} \int_0^{l(n)} (j \vee sl(n)) \, dj \\ &= 2s^{-1}l(n) + s^{-1} \cdot \frac{1}{2} (l(n)^2 + (sl(n))^2) \leq \frac{1+s^2}{2s} l(n)(l(n) + 4). \end{aligned}$$

Now, we use the equalities  $\deg(V_n) = 2\#E_n + \#\partial V_n$  and  $\#E_n = l(n)(l(n) - 1)$  to infer

$$\frac{\#\partial V_n}{\text{vol}(V_n)} = 1 - \frac{2\#E_n}{\text{vol}(V_n)} \leq 1 - \frac{2s}{(1+s^2)} \frac{(l(n) - 1)}{(l(n) + 4)}$$

which by the discussion given at the beginning implies  $\alpha_\infty = 1 - 2s/(1 + s^2)$ .

### 3. EXPONENTIAL VOLUME GROWTH

In this section we prove Theorem 1.5. This is followed by an example which shows sharpness of the bound.

*Proof of Theorem 1.5.* Let  $s \in (r_\infty, 1)$  and let  $n$  be chosen according to Lemma 2.1. For  $v \in V$ , let  $k \geq 0$  be such that  $\#B_k(v) \geq 2n$ . We use Lemma 2.1 to estimate

$$\begin{aligned} \text{vol}(B_k(v)) &= \sum_{w \in B_k(v)} \deg(w) \geq \sum_{j=0}^{\#B_k(v)-1} \deg(v_j) \geq \frac{1}{s} \sum_{j=n}^{\#B_k(v)-1} (j+1) \\ &\geq \frac{1}{s} \int_{\frac{1}{2}\#B_k(v)}^{\#B_k(v)} j \, dj = \frac{3}{8s} (\#B_k(v))^2. \end{aligned}$$

Next, we estimate  $\#B_k$ . By Lemma 2.1, there is a  $w \in B_k(v)$  such that  $\deg(w) \geq s^{-1}\#B_k(v)$ . We obtain  $w \notin B_{k-1}(v)$  since else  $B_1(w) \subseteq B_k(v)$  which is a contradiction to  $s < 1$ . Hence,

$$B_1(w) \dot{\cup} B_{k-2}(v) \subseteq B_{k+1}(v)$$

and consequently,

$$\#B_{k+1}(v) \geq \#B_1(w) + \#B_{k-2}(v) \geq s^{-1}\#B_k(v) + \#B_{k-2}(v).$$

Hence, a lower bound of the growth of  $b_k = \#B_k(v)$  is encoded in the eigenvalues of the matrix

$$M_s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & s^{-1} \end{pmatrix}.$$

Indeed, the inequality above translates into the componentwise vector inequality  $(b_{k-1}, b_k, b_{k+1})^T \geq M_s (b_{k-2}, b_{k-1}, b_k)^T$ . The characteristic polynomial  $p$  of  $M_s$  is given by  $p(z) = z^3 - s^{-1}z^2 - 1$  with the largest real root  $a_s$ . By the Perron-Frobenius theorem the eigenvector of the irreducible matrix  $M_s$  to the eigenvalue  $a_s$  has strictly positive entries. Thus, there is  $C_s > 0$  such that for  $k$  large enough, we have  $\#B_k(v) \geq C_s a_s^k$ . By using the estimate  $\text{vol}(B_k(v)) \geq \frac{3}{8s} (\#B_k(v))^2$ , we obtain

$$\text{vol}(B_k(v)) \geq D_s a_s^{2k}$$

for some  $D_s > 0$  and large enough  $k$ . This finishes the proof since  $a_s$  is continuous in  $s$  for  $s > 0$  and  $a = a_{r_\infty}$ .  $\square$

Below we give an example for a family of exponentially growing graphs with their values for  $r_\infty$  being arbitrarily close to zero. These graphs are so called antitrees, see e.g. [KLW13, Woj11] and the examples show qualitatively that the above theorem is sharp in the sense of exponential growth.

**Example 3.1.** For  $\sigma \in \mathbb{N}$  with  $\sigma \geq 2$ , let a graph  $G_\sigma = (V_\sigma, E_\sigma)$  be given with  $V = \bigcup_{n \geq 0} S_n$  such that  $\#S_n = \sigma^n$ . Moreover, every vertex in  $S_n$  is connected to every vertex except to itself in  $S_{n-1} \cup S_n \cup S_{n+1}$ ,  $n \geq 1$ . Hence, for all  $n \geq 2$ , we have  $v \in S_n$  if and only if  $\deg v = \sigma^{n-1} + \sigma^n + \sigma^{n+1} - 1$ . Let  $k \geq \sigma^2 + \sigma$  be an integer and  $n \geq 2$  is the unique integer such that

$$\sigma^{n-2} + \sigma^{n-1} + \sigma^n - 1 \leq k < \sigma^{n-1} + \sigma^n + \sigma^{n+1} - 1.$$

Then,

$$\frac{\eta(k)}{k} \leq \frac{\eta(\sigma^{n-2} + \sigma^{n-1} + \sigma^n - 1)}{\sigma^{n-2} + \sigma^{n-1} + \sigma^n - 1} = \frac{\sum_{j=0}^{n-1} \sigma^j}{\sigma^{n-2} + \sigma^{n-1} + \sigma^n - 1} < \frac{1}{\sigma - 1}.$$

This shows that  $r_{\infty, \sigma} := \limsup_{k \rightarrow \infty} \eta(k)/k$  is indeed strictly smaller than one for each  $\sigma \geq 2$ . We also derive from the above computation that  $\lim_{\sigma \rightarrow \infty} r_{\infty, \sigma} = 0$ .

There are also examples that show that the precise bound is actually sharp. However, the construction is rather lengthy, so, we refrain from giving the details.

## 4. STOCHASTIC INCOMPLETENESS

This section is devoted to the proof of Theorem 1.6 which shows that rapidly branching graphs with large growth acceleration are stochastically incomplete.

*Proof of Theorem 1.6.* By [KL10, Theorem 25, Proposition 28] (cf. also [Woj08, Woj09]) stochastic incompleteness is equivalent to existence of a bounded and positive  $\lambda$ -subharmonic function  $u$  for some  $\lambda > 0$ , i.e.,  $u > 0$  satisfies

$$\sum_{w \sim v} (u(v) - u(w)) + \lambda u(v) \leq 0, \quad v \in V.$$

Indeed, it suffices for  $u$  to be  $\lambda$ -subharmonic outside of a finite set, see [KL12, Corollary 1.2] or [Hua11b, Theorem 4.1].

We define for  $p \in (0, 1)$ ,  $\lambda > 0$ , the function  $u : V \rightarrow (0, 1)$  by

$$u(v) = 1 - (\deg(v) + \lambda)^{-p}, \quad v \in V.$$

We observe

$$\sum_{w \sim v} (u(v) - u(w)) + \lambda u(v) = \lambda - (\deg(v) + \lambda)^{1-p} + \sum_{w \sim v} (\deg(w) + \lambda)^{-p}.$$

We proceed by estimating the third term on the right hand side. Let  $r_\infty < s < e^{-1}$  and let  $n$  be chosen according to Lemma 2.1. Below we will choose  $\lambda$  and  $p$  such that the function  $u$  becomes  $\lambda$ -subharmonic outside of the set  $K = \bigcup_{k=0}^{n-1} B_1(v_k)$  which is the set of neighbors of  $\{v_0, \dots, v_{n-1}\}$ . For  $v \in V \setminus K$ , we find using Lemma 2.1

$$\begin{aligned} \sum_{w \sim v} (\deg(w) + \lambda)^{-p} &\leq \sum_{k=n}^{n+\deg(v)-1} (\deg(v_k) + \lambda)^{-p} \leq \sum_{k=n}^{n+\deg(v)-1} \left( \frac{(k+1)}{s} + \lambda \right)^{-p} \\ &\leq s^p \sum_{k=1}^{\deg(v)} (k + s\lambda)^{-p} \leq s^p \int_0^{\deg(v)} (k + s\lambda)^{-p} dk \\ &\leq \frac{s^p}{1-p} \left( (\deg(v) + \lambda)^{1-p} - (s\lambda)^{1-p} \right), \end{aligned}$$

where we recall  $s < 1$  and  $p < 1$  for the last estimate. Since we have  $\lim_{p \rightarrow 0} (1-p)^{1/p} = e^{-1}$  and  $r_\infty < s < e^{-1}$ , there is  $p$  such that  $s \leq (1-p)^{1/p}$ . We fix this choice of  $p$  for what follows and remark

$$\frac{s^p}{1-p} \leq 1.$$

Furthermore, we let  $\lambda$  be chosen such that  $\lambda^p \leq s/(1-p)$ . This yields

$$\lambda \leq \frac{s}{1-p} \lambda^{1-p} = \frac{s^p}{1-p} (s\lambda)^{1-p}.$$

Putting together what we have estimated so far with the equality in the beginning we find that

$$\begin{aligned} & \sum_{w \sim v} (u(v) - u(w)) + \lambda u(v) \\ & \leq \lambda - (\deg(v) + \lambda)^{1-p} + \frac{s^p}{1-p} (\deg(v) + \lambda)^{1-p} - \frac{s^p}{1-p} (s\lambda)^{1-p} \\ & \leq 0. \end{aligned}$$

According to the discussion in the beginning this finishes the proof.  $\square$

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